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Linearized Dynamic Equations for Spacecraft Subject to J_2 Perturbations

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Introduction

THE dynamic equations of motion for a spacecraft relative to a neighboring orbit have taken on added importance in the last few years due to the possibility of spacecraft orbiting in formation. For a number of mathematical and physical reasons, linearized dynamic equations are of interest. The survey by Carter[1] shows that there are a number of ways to linearize the equations of motion for a spacecraft. Of particular interest is the so-called Hill's equation[2] a somewhat unfortunate name because "Hill's equation is a convenient abbreviation defining the class of homogeneous, linear, second-order differential equations with real, periodic coefficients."[3] For this and other reasons, the linearized orbital equations are more commonly known as the Clohessy–Wiltshire (C–W) equations.[4] In any case, these equations are typically used for an inverse-square gravity field, although the inclusion of a perturbation acceleration allows them to be used, in principle, for more general cases. As shown by Alfriend et al.[5] and others, the effect of a J_2 perturbation gradually destroys the configuration of the formation of the orbiting spacecraft. Consequently, a number of propellant-reduction formation-keeping strategies are proposed as summarized by Vadalli et al.[6] To better understand the effects of J_2 and eccentricity, Gim and Alfriend[7] obtain state transition matrices by way of various similarity transformations. From this brief review of the literature, it is apparent that a set of linearized dynamic equations that can handle the effects of J_2 would be useful in further analyzing the dynamics of spacecraft in formation.

In this Note, we present a new set of linearized dynamic equations that account for the J_2 perturbations. We limit the results for the case of a circular reference orbit. In principle, these equations can be generalized for elliptical reference orbits as described by Melton.[8]

Main Result

We present the main result here and derive it in the following section. For clarity, we define a reference frame O as one that is centered at a radial distance R_0 from the center of the Earth, whose origin moves at a circular speed $[=\sqrt{(\mu/R_0)}]$, while the frame rotates at a circular angular velocity $[=n=\sqrt{(\mu/R_0^3)}]$ about its number 3 axis such that the number 1 axis is along the outward R_0 direction. In other words, this frame is equivalent to a Frenet frame associated with a hypothetical spacecraft orbiting Earth in an inverse-square gravity field, that is, with its J_2 component removed. Let, $\boldsymbol{r}$ be the position vector of a (real) spacecraft in this coordinate system. When $\dot{\boldsymbol{r}}$ and $\ddot{\boldsymbol{r}}$ are the first and second time derivatives of $\boldsymbol{r}$ in O, the linearized equations of motion with J_2 effects can be written as

$$\ddot{\boldsymbol{r}} + C\dot{\boldsymbol{r}} + [K + P(t)]\boldsymbol{r} + \boldsymbol{q}(t) = \boldsymbol{a} \tag{1}$$

where $\boldsymbol{a}$ is the perturbing acceleration on the spacecraft (resulting, for example, from the higher-order gravitational harmonics, thrust, drag, etc.) and the other quantities are defined as follows: C and K are 3×3 constant matrices,

$$C = 2n\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad K = n^2\begin{bmatrix} -3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$P(t)$ is a 3×3 symmetric matrix periodic function that can be written as

$$P(t) = n^2 J_R \begin{bmatrix} 12\sin^2 i\,\sin^2(nt) - 4 & -4\sin^2 i\,\sin(2nt) & -4\sin(2i)\sin(nt) \\ -4\sin^2 i\,\sin(2nt) & 1+\sin^2 i\,(2-7\sin^2(nt)) & \sin(2i)\cos(nt) \\ -4\sin(2i)\sin(nt) & \sin(2i)\cos(nt) & 3-\sin^2 i\,(2+5\sin^2(nt)) \end{bmatrix} \tag{3}$$

where i is the inclination of the hypothetical spacecraft, t is the time computed from passage of the ascending node, that is, the clock starts when the hypothetical spacecraft crosses the ascending node, and J_R is a nondimensional number given by

$$J_R = \frac{3J_2R_\oplus^2}{2R_0^2} \tag{4}$$

Last, $\boldsymbol{q}(t)$ is the 3×1 vector periodic function

$$\boldsymbol{q}(t) = n^2 J_R R_0 \begin{bmatrix} 1 - 3\sin^2(nt)\sin^2 i \\ \sin(2nt)\sin^2 i \\ \sin(nt)\sin(2i) \end{bmatrix} \tag{5}$$

Remark 1: It is obvious that for $J_2 = 0$, Eq. (1) is the familiar C–W equation. For $J_2 \neq 0$, Eq. (1) belongs to the class of Hill's equation (see Ref. 3).

Remark 2: The equations are deceptive in the sense that it appears that the J_2-perturbed motion of the spacecraft does not depend on Ω, the right ascension of the ascending node. This interpretation is not true because the equations of motion of the spacecraft are written in a frame that is deliberately chosen to be unaffected by J_2. That is, the orbital elements in Eq. (1) are that of the static reference orbit and not equal to the dynamic (osculating) elements of the spacecraft except possibly at the time of initialization.

Remark 3: The simplest case is zero inclination. In this situation, $\boldsymbol{q}(t)$ is a constant with its last two elements being zero, whereas the P matrix simplifies to a constant diagonal matrix. Note that these equations are not singular for zero inclination.

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Remark 4: Equations for relative motion between two real spacecraft follow directly from Eq. (1) with $\boldsymbol{q}(t)$ set to zero. In this case, $\boldsymbol{r}$ is to be interpreted as the position vector between the two real spacecraft, but the derivatives are still in the O frame. This follows quite simply by setting $\boldsymbol{r} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and writing the dynamic equations for each spacecraft using Eq. (1). Also, $\boldsymbol{a}$ now denotes the differential perturbing acceleration $\boldsymbol{a} = \boldsymbol{a}_2 - \boldsymbol{a}_1$.

Outline of the Derivation of the Main Result

A derivation of the equations of motion from first principles is quite lengthy; hence, we make extensive use of prior results but provide sufficient details for easy reproduction.

Let $\boldsymbol{r} = (x, y, z)$ be the components of the position vector of a (real) spacecraft in the orbital frame. Then, the equations of motion for the spacecraft can be shown to be[2]

$$\ddot{x} - 2n\dot{y} - 3n^2 x = a_x + p_x \tag{6a}$$

$$\ddot{y} + 2n\dot{x} = a_y + p_y \tag{6b}$$

$$\ddot{z} + n^2 z = a_z + p_z \tag{6c}$$

where p_x, p_y, and p_z are acceleration perturbations (along the appropriate directions) resulting from J_2 and all other terms are as defined earlier. Thus, the derivation reduces to expressing these J_2-perturbing accelerations in terms of the other chosen variables. From Eq. (8.28) of Ref. 2, it follows that the J_2-perturbation acceleration can be written as

$$\boldsymbol{p} = \frac{3J_2\mu R_\oplus^2}{2R^5}\left[\left(5\frac{Z^2}{R^2} - 1\right)(X\boldsymbol{n}_1 + Y\boldsymbol{n}_2) + Z\left(5\frac{Z^2}{R^2} - 3\right)\boldsymbol{n}_3\right] \tag{7}$$

where $\boldsymbol{n}_1$, $\boldsymbol{n}_2$, and $\boldsymbol{n}_3$ define the standard geocentric inertial (GCI) frame N, with respect to which the standard orbital elements of a spacecraft are defined. The quantities X, Y, and Z are the components of the position vector of the spacecraft in GCI, and R is its magnitude, $R = \sqrt{(X^2 + Y^2 + Z^2)}$.

The direction cosine matrix (DCM) between the orbital frame and GCI can be obtained from a body 3–1–3 rotation of GCI with the sequence of angular rotations Ω, i, and nt, respectively. Note that Ω, i, and n are the constant orbital elements of the hypothetical spacecraft. Following the notation of Kane et al.,[9] but using transposed matrices, the DCM is given by

$$^{O}\boldsymbol{C}^{N} = \begin{bmatrix} \cos nt\cos\Omega - \sin nt\cos i\sin\Omega & \cos nt\sin\Omega + \sin nt\cos i\cos\Omega & \sin nt\sin i \\ -\sin nt\cos\Omega - \cos nt\cos i\sin\Omega & -\sin nt\sin\Omega + \cos nt\cos i\cos\Omega & \cos nt\sin i \\ \sin i\sin\Omega & -\sin i\cos\Omega & \cos i \end{bmatrix} \tag{8}$$

Thus, we can find p_x, p_y, and p_z from the transformation

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = {}^{O}\boldsymbol{C}^{N}\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} \tag{9}$$

where p_1, p_2, and p_3 are the components of $\boldsymbol{p}$ in GCI [see Eq. (7)]. Similarly, X, Y, and Z may also be transformed to x, y, and z from

$$\begin{bmatrix} X \\ Y \\ Z \end{bmatrix} = {}^{N}\boldsymbol{C}^{O}\begin{bmatrix} x + R_0 \\ y \\ z \end{bmatrix} \tag{10}$$

Substituting Eqs. (8–10) in Eq. (7) yields the required components of the J_2 perturbations in the orbital frame. These components are nonlinear in the x, y, and z variables. Because we desire only the linear terms, the components can be linearized by a series expansion of the lowest common denominator of Eq. (7), namely, R^7

$$1/R^7 = 1\Big/(X^2 + Y^2 + Z^2)^{\frac{7}{2}} \tag{11a}$$

$$1/R^7 = 1\Big/\left(x^2 + y^2 + z^2 + R_0^2 + 2xR_0\right)^{\frac{7}{2}} \tag{11b}$$

$$1/R^7 = (1 - 7x/R_0)\Big/R_0^7 + \mathrm{HOT} \tag{11c}$$

where higher-order terms (HOT) in x, y, and z are a part of Eq. (11c). Substituting Eqs. (7), (8), (10), and (11) in Eq. (9) and simplifying is an extremely laborious process. Hence, the symbolic manipulator, Maple[10] was used for many of the steps to simplify the algebra. This results in

$$p_x \cong -\frac{3J_2n^2R_\oplus^2}{2R_0}\left[\left(1 - \frac{7x}{R_0}\right)(1 - 3\sin^2 nt\sin^2 i) + \frac{x}{R_0}(9\cos^2 nt + 9\cos^2 i\sin^2 nt - 6) + \frac{y}{R_0}(8\sin nt\cos nt\cos^2 i - 8\sin nt\cos nt) - \frac{z}{R_0}(8\sin nt\sin i\cos i)\right] \tag{12a}$$

$$p_y \cong -\frac{3J_2n^2R_\oplus^2}{2R_0}\left[\left(1 - \frac{7x}{R_0}\right)\sin(2nt)\sin^2 i + \frac{x}{R_0}(6\sin nt\cos nt\sin^2 i) + \frac{y}{R_0}(7\cos^2 nt + 5\cos^2 i - 4 - 7\cos^2 nt\cos^2 i) + \frac{z}{R_0}(2\cos nt\sin i\cos i)\right] \tag{12b}$$

$$p_z \cong -\frac{3J_2n^2R_\oplus^2}{2R_0}\left[\left(1 - \frac{7x}{R_0}\right)\sin nt\sin(2i) + \frac{x}{R_0}(6\cos i\sin i\sin nt) + \frac{y}{R_0}(2\cos nt\sin i\cos i) \; \frac{z}{R_0}(7\cos^2 i + 5\cos^2 nt - 4 - 5\cos^2 nt\cos^2 i)\right] \tag{12c}$$

where the approximations imply that we have dropped all the nonlinear terms in x, y, and z. Substituting Eqs. (12) in Eq. (6) and simplifying yields Eq (1).

Conclusions

A set of linearized dynamic equations for an orbiting spacecraft subject to J_2 perturbations has been derived. These equations are singularity free and hold for all orbital inclinations. Because linearized equations should not be used for long-term prediction, these equations are valid for analyzing short-term dynamics. In principle, they can be applied to analyze the relative motion of two spacecraft with small eccentricities by dropping the periodic forcing function. In any case, the retention of generic perturbing accelerations in these equations makes them useful for analyzing the dynamics and control of spacecraft orbiting in formation.

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Spacecraft Vibration Reduction Following Thruster Firing for Orbit Adjustment

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Introduction

WAITING for the decay of vibrations caused by thruster firing reduces time for data taking in spacecraft operations. This Note is concerned with vibration reduction following thruster firing for orbit adjustment. The principle used for vibration reduction in a robust manner is input shaping,[1] which was originally proposed for command shaping in distinct steps and was demonstrated during the shuttle manipulator arm slewing. An equivalent technique implemented in terms of time-delay control was established in Ref. 2. The method of Ref. 1 was extended to constant magnitude on–off thrusters in Ref. 3. Input shaping is an effective and easy-to-implement technique that has been used in disk drive positioning,[4] flexible robot arm control,[5] and many other applications and has been proposed as a baseline design feature for slewing and momentum dumping of the Next Generation Space Telescope.[6]

In this Note we describe two ways of shaping of a thrust–time profile for vibration suppression. The first method uses constant magnitude on–off thrust, where the thruster switching times are computed by solving the input shaping equations of Ref. 1 together with the equation for thrust impulse to be imparted. The second approach uses a two-level thrust by convolution of the thrust pulse with the three-impulse sequence of Ref. 1, and then scales up the torque by the method of Ref. 7. In both cases, input shaping is done for one dominant mode that is excited by the closed-loop control. A detailed nonlinear flexible multibody dynamics model with many appendage vibration modes is then used for simulation of the spacecraft, based on a code with the dynamics formulation of Ref. 8, to verify the effectiveness of vibration suppression.

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Table 1 Timed impulse sequence for constant magnitude on off thrust

t_i	A_i
0	1
t_1	−1
t_2	1
t_3	−1
——	——
t_{n-1}	1
t_n	−1

Vibration Reduction Following Thruster Firing

On–off thrust of constant magnitude can be seen as the result of a convolution of a step with the sequence of impulses given in Table 1.

Singer and Seering[1] derived the basic result of robust vibration suppression by applying a sequence of n impulses on a vibratory system. For the timed sequence of impulses of Table 1, the equations for zero residual vibration for the jth mode become

$$\sum_{i=1}^{n}(-1)^i \sin\omega_j t_i = 0, \qquad j = 1, 2, \ldots \tag{1}$$

$$1 + \sum_{i=1}^{n}(-1)^i \cos\omega_j t_i = 0, \qquad j = 1, 2, \ldots \tag{2}$$

Robustness to frequency modeling error gives rise to the equations

$$\sum_{i=1}^{n}(-1)^i t_i \sin\omega_j t_i = 0, \qquad j = 1, 2, \ldots \tag{3}$$

$$\sum_{i=1}^{n}(-1)^i t_i \cos\omega_j t_i = 0, \qquad j = 1, 2, \ldots \tag{4}$$

The requirement of net impulse I for orbit adjustment from thrust of magnitude F translates to the condition

$$F(t_1 + t_3 - t_2 + \cdots + t_n - t_{n-1}) = I \tag{5}$$

For a single mode to suppress, Eqs. (1–5) have five unknown times to solve for three pulses starting at $t = 0$. For two modes to suppress, these represent nine equations in nine unknown times, the solution of which provides five pulses. In general, we see that there is an exact solution possible for the $4n + 1$ impulse times suppressing n modes and meeting the net impulse requirement by $(2n + 1)$ timed pulses.

An alternative to the preceding approach is the scaled convolution method, which scales and successively convolves the thrust–time history with the three-impulse sequence[1] for each vibration mode of period T_i, $i = 1, \ldots, n$:

$$\begin{aligned} F(t) = \alpha F[u(t) - u(t-\tau)] &* [0.25\delta(t) + 0.5\delta(t - T_1/2) \\ &+ 0.25\delta(t - T_1)] * \cdots * [0.25\delta(t) + 0.5\delta(t - T_n/2) \\ &+ 0.25\delta(t - T_n)] \end{aligned} \tag{6}$$

where F is the maximum thruster force; $u(t)$ is the unit step function; $u(t-\tau)$ is the delayed step function, where τ is the unshaped thruster duration; $\delta(t)$ is the impulse function; and $\delta(t - T_i)$ is the impulse function delayed by T_i, the period of vibration of the ith mode to be suppressed. In Eq. (6), α is a scale factor; normally, if the unshaped thruster pulse duration is greater than the period of vibration, α is unity. However, if the nominal thruster firing time